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On the representation theory of orthofermions and orthosupersymmetric realization of parasupersymmetry and fractional supersymmetry

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Abstract

We construct a canonical irreducible representation for the orthofermion algebra of arbitrary order, and show that every representation decomposes into irreducible representations that are isomorphic to either the canonical representation or the trivial representation. We use these results to show that every orthosupersymmetric system of order p has a parasupersymmetry of order p and a fractional supersymmetry of order $p + 1$.

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1. Introduction

Orthofermions were originally introduced by Khare *et al* [1] in an attempt to obtain a generalization of supersymmetry called orthosupersymmetry. Recently [2], it has been realized that orthofermions may be used to construct parafermions of order 2, and that every orthosupersymmetric system possesses topological symmetries [3]. In particular, given an orthosupersymmetric system of order p, one can construct a fractional supersymmetric system of order $p + 1$ [2]. The main ingredient leading to these observations is the algebra of orthosupersymmetric quantum mechanics [1]:

$$[H, Q_\alpha] = 0 \tag{1}$$

$$Q_\alpha Q_\beta^\dagger + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} Q_\gamma^\dagger Q_\gamma = 2\delta_{\alpha\beta} H \tag{2}$$

$$Q_\alpha Q_\beta = 0 \tag{3}$$

where Q_α are the generators of the orthosupersymmetry, $\alpha, \beta \in \{1, 2, \ldots, p\}$ and $\delta_{\alpha\beta}$ stands for the Kronecker delta function. The simplest quantum system possessing orthosupersymmetry of order p is a system with bosonic and orthofermionic degrees of freedom. For this system the generators of orthosupersymmetry are given by $Q_\alpha = \sqrt{2}a^\dagger c_\alpha$ where a is the annihilation operator for a boson and c_α are the annihilation operators for orthofermions of order p. They are defined through the relations [1]

$$[a, a^\dagger] = 1 \qquad [a, c_\alpha] = [a, c_\alpha^\dagger] = 0$$
$$c_\alpha c_\beta^\dagger + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} c_\gamma^\dagger c_\gamma = \delta_{\alpha\beta} 1 \tag{4}$$
$$c_\alpha c_\beta = 0 \tag{5}$$

where 1 stands for the identity operator. The study of orthosupersymmetry [1] relies on a matrix representation of orthofermions of order p where c_α are represented by $(p+1)\times(p+1)$ matrices with entries

$$[c_\alpha]_{ij} = \delta_{i,1}\delta_{j,\alpha+1} \qquad \forall i, j \in \{1, \ldots, p+1\}. \tag{6}$$

The purpose of this paper is to study the general representations of the orthofermion algebra, i.e. equations (4) and (5), and to explore the implications of this study for ortho-, para- and fractional supersymmetry of arbitrary order.

The organization of the paper is as follows. In section 2, we construct a canonical irreducible representation for the orthofermion algebra. In section 3, we examine general representations of the orthofermion algebra, and show that every representation decomposes into the irreducible representations that are either isomorphic to the canonical representation or the trivial representation. In section 4, we construct the ladder operators for the canonical representation and derive some of their basic properties. In section 5, we use the results of the preceding sections to show that every orthosupersymmetric system of order p possesses a parasupersymmetry of order p and a fractional supersymmetry of order $p+1$. In sections 6, we summarize our results and present our concluding remarks.

2. The canonical irreducible representation of the orthofermion algebra

We begin our analysis of the orthofermion algebra, i.e. equations (4) and (5), by introducing

$$\Pi := 1 - \sum_{\alpha=1}^{p} c_\alpha^\dagger c_\alpha. \tag{7}$$

This allows us to write equation (4) in the form

$$c_\alpha c_\beta^\dagger = \delta_{\alpha\beta}\Pi. \tag{8}$$

It is not difficult to show that Π is a Hermitian projection operator:

$$\Pi^2 = \Pi = \Pi^\dagger. \tag{9}$$

This follows from equations (5), (7) and (8). Furthermore, for all $\alpha \in \{1, 2, \ldots, p\}$,

$$\Pi\, c_\alpha = c_\alpha \qquad c_\alpha^\dagger \Pi = c_\alpha^\dagger \tag{10}$$
$$c_\alpha \Pi = 0 \qquad \Pi\, c_\alpha^\dagger = 0. \tag{11}$$

Next, let $\mathcal{A}$ denote the (complex associative $*$) algebra generated by $2p$ generators: $c_\alpha, c_\alpha^\dagger$ with $\alpha \in \{1, 2, \ldots, p\}$, and subject to relations (5), (7) and (8)[1]. Then in view of these relations and equations (10) and (11), elements of $\mathcal{A}$ have the general form

$$x = \lambda\Pi + \sum_{\alpha=1}^{p}(\nu_\alpha c_\alpha + \mu_\alpha c_\alpha^\dagger) + \sum_{\alpha,\beta=1}^{p} \sigma_{\alpha\beta}\, c_\alpha^\dagger c_\beta \tag{12}$$

[1] Clearly, the $*$ operation is given by $\dagger$.

where λ, ν_α, μ_α and $\sigma_{\alpha\beta}$ are complex numbers. As seen from equation (12), $\mathcal{A}$ is a $(p+1)^2$-dimensional complex vector space. We can use this vector space as a representation space for orthofermion algebra. However, as we shall see in section 4 this would lead to a reducible representation. Therefore, we will restrict to a subrepresentation.

Let $\mathcal{A}_0 \subset \mathcal{A}$ be the span of Π and $c_\alpha^\dagger$ and

$$x_0 := \lambda\Pi + \sum_{\alpha=1}^{p} \mu_\alpha c_\alpha^\dagger$$

be an arbitrary element of $\mathcal{A}_0$. Then, in view of equation (8) and (10), for all $\alpha \in \{1, 2, \ldots, p\}$,

$$c_\alpha x_0 = \mu_\alpha \Pi \qquad c_\alpha^\dagger x_0 = \lambda c_\alpha^\dagger. \tag{13}$$

These equations suggest that $\mathcal{A}_0$ is the representation space for a Fock space representation of orthofermions. Following the standard notation, we set

$$|0\rangle := \Pi \qquad \text{and} \qquad \forall\alpha \in \{1, 2, \ldots, p\} \qquad |\alpha\rangle := c_\alpha^\dagger. \tag{14}$$

This yields

$$\forall\alpha \in \{1, 2, \ldots, p\} \qquad c_\alpha|0\rangle = 0 \qquad \text{and} \qquad |\alpha\rangle = c_\alpha^\dagger|0\rangle. \tag{15}$$

Furthermore, using equation (11) we have

$$\Pi|0\rangle = |0\rangle \qquad \text{and} \qquad \forall\alpha \in \{1, 2, \ldots, p\} \qquad \Pi|\alpha\rangle = 0. \tag{16}$$

Therefore, Π is the projection onto the 'vacuum' state vector $|0\rangle$.

As a vector space $\mathcal{A}_0$ is isomorphic to $\mathbb{C}^{p+1}$. The vectors $|n\rangle$ with $n \in \{0, 1, \ldots, p\}$ form a basis for $\mathcal{A}_0$. In this basis, $|n\rangle$ may be identified with column vectors whose kth component is given by δ_{nk} and the operators c_α are represented by $(p+1) \times (p+1)$ matrices with entries

$$[c_\alpha]_{ij} = \delta_{i,1}\delta_{j,\alpha+1}. \tag{17}$$

This is precisely the matrix representation (6) of [1]. Note that unlike in [1], here we construct the representation space: $\mathcal{A}_0$. Since, we have obtained the action of c_α and $c_\alpha^\dagger$ on the basis vectors $|n\rangle$, we can represent all the elements of $\mathcal{A}$ by linear operators (endomorphisms) mapping $\mathcal{A}_0$ into itself, i.e. we have a representation $\rho_0 : \mathcal{A} \to \mathrm{End}(\mathcal{A}_0)$ of the algebra $\mathcal{A}$. Here 'End' abbreviates the 'space of endomorphisms of', and by a representation $\rho : \mathcal{A} \to \mathrm{End}(V)$ in a complex vector space V we mean a linear map satisfying

$$\rho(x_1x_2) = \rho(x_1)\rho(x_2) \qquad \forall x_1, x_2 \in \mathcal{A}. \tag{18}$$

We shall also postulate

$$\forall x \in \mathcal{A} \qquad \rho(x^\dagger) = \rho(x)^\dagger \tag{19}$$

if V is endowed with an inner product.

Note that the representation ρ_0 is an irreducible representation. This may be easily verified by inspecting the matrices (17).

3. Representation theory of the orthofermion algebra

Let V be an arbitrary complex vector space, $\rho : \mathcal{A} \to \mathrm{End}(V)$ be a representation of $\mathcal{A}$, and V_0 be the subspace of V defined by

$$V_0 := \mathrm{Im}\,(\rho(\Pi)) := \{\rho(\Pi)v \mid v \in V\}.$$

Lemma 1. *If $V_0 = \{0\}$, then ρ is a trivial representation, i.e. for all $x \in \mathcal{A}$ and $v \in V$, $\rho(x)v = 0$.*

Proof. In view of equation (18), it is sufficient to prove that $\rho(c_\alpha)v = \rho(c_\alpha^\dagger)v = 0$. Because $V_0 = \{0\}$, for all $u \in V$, $\rho(\Pi)u = 0$. But then according to equations (18), (10) and (11),

$$\rho(c_\alpha)v = \rho(\Pi\, c_\alpha)v = \rho(\Pi)[\rho(c_\alpha)v] = 0 \qquad \rho(c_\alpha^\dagger)v = \rho(c_\alpha^\dagger\Pi)v = \rho(c_\alpha^\dagger)[\rho(\Pi)v] = 0.$$

□

Now, suppose that ρ is not a trivial representation. Then V_0 is a nontrivial subspace of V. Let $\{e_i\}$ be a basis of V_0 and V_i be the subspaces of V defined by

$$V_i := \mathrm{Span}\left(\{e_i\}\bigcup\{\rho(c_\alpha^\dagger)e_i\}_{\alpha\in\{1,2,\cdot s,p\}}\right).$$

Then we can prove the following lemma.

Lemma 2. *For all $x \in \mathcal{A}$, $\rho(x)$ maps V_i into V_i.*

Proof. It suffices to show that for all $\alpha \in \{1, 2, \ldots, p\}$, $\rho(c_\alpha)$ and $\rho(c_\alpha^\dagger)$ map V_i into V_i. Let $v \in V_i$, then there are complex numbers μ_α and λ such that $v = \sum_{\beta=1}^p \mu_\beta\rho(c_\beta^\dagger)e_i + \lambda e_i$. In view of equations (18), (11), (9), and $e_{i_\ell} \in V_\ell$,

$$\rho(c_\alpha)v = \sum_{\alpha=1}^p \mu_\beta\rho(c_\alpha c_\beta^\dagger)e_i + \lambda\rho(c_\alpha)e_i = \mu_\alpha e_i \in V_i$$

$$\rho(c_\alpha^\dagger)v = \sum_{\alpha=1}^p \mu_\beta\rho(c_\alpha^\dagger c_\beta^\dagger)e_i + \lambda\rho(c_\alpha^\dagger)e_i = \lambda\rho(c_\alpha^\dagger)e_i \in V_i.$$

□

A direct implication of lemma 2 is that for each basis vector e_i of V_0 the restriction

$$\rho_i = \rho|_{V_i} : \mathcal{A} \to \mathrm{End}(V_i)$$

of ρ provides a representation of the algebra $\mathcal{A}$.

Furthermore, introducing

$$|0\rangle_i := e_i \qquad \text{and} \qquad |\alpha\rangle_i := \rho_i(c_\alpha^\dagger)|0\rangle_i \qquad \forall\alpha \in \{1, 2, \ldots, p\}$$

we can easily show that $|n\rangle_i$, with $n \in \{0, 1, \ldots, p\}$, are basis vectors for V_i and that in this basis the operators $\rho_i(c_\alpha)$ are represented by matrices whose entries are given by the right-hand side of equation (6). Therefore, the representations ρ_i are equivalent to the representation ρ_0. In particular, they are irreducible representations.

Next, consider the case where the dimension of V_0 is greater than one.

Lemma 3. *Let e_{i_1} and e_{i_2} be distinct basis vectors of V_0. Then $V_{i_1} \cap V_{i_2} = \{0\}$.*

Proof. Suppose $v \in V_{i_1} \cap V_{i_2}$. Then there are complex numbers μ_α, μ'_α, λ and λ' such that

$$v = \sum_{\alpha=1}^p \mu_\alpha\rho(c_\alpha^\dagger)e_{i_1} + \lambda e_{i_1} = \sum_{\alpha=1}^p \mu'_\alpha\rho(c_\alpha^\dagger)e_{i_2} + \lambda' e_{i_2}. \tag{20}$$

Applying $\rho(\Pi)$ to both sides of the second equation in (20) and using (18), (11), (9), and $e_{i_\ell} \in V_\ell$, we find $\lambda e_{i_1} = \lambda' e_{i_2}$, which implies $\lambda = \lambda' = 0$. Similarly, applying $\rho(c_\beta)$ to both sides of (20) for an arbitrary $\beta \in \{1, 2\ldots, p\}$, we have $\mu_\beta e_{i_1} = \mu'_\beta e_{i_2}$ which yields $\mu_\beta = \mu'_\beta = 0$. Hence, $v = 0$. □

Now, we are in a position to address the issue of the decomposition of an arbitrary representation ρ into irreducible representations. The algebra $\mathcal{A}$ does not contain a unit 1. In the following we shall extend $\mathcal{A}$ by adding 1 as a generator satisfying: $\forall x \in \mathcal{A}$, $1x = x1 = x$. Inclusion of 1 allows us to use equation (7) in the representations of $\mathcal{A}$. Clearly, we have $\forall x \in \mathcal{A}, \rho(1)\rho(x) = \rho(x)\rho(1) = \rho(x)$. For the representations ρ_i, we have $\rho_i(1) = I_i$, where I_i is the identity operator acting on V_i. This follows from the equivalence of ρ_i and ρ_0. Note also that for a trivial representation, we have $\rho(1) = 0$.

Lemma 4. *Suppose that $\rho(1) = I$, where I denotes the identity operator acting on V, and let $V_* := \oplus_i V_i$. Then $V = V_*$.*

Proof. Let $w \in V - V_*$, in particular $w \neq 0$ and $e := \rho(\Pi)w \in V_0 \subset V_*$, so that $e \in V_*$ and $e \neq w$. Now, using equation (7), we have

$$\sum_{\alpha=1}^{p} \rho(c_\alpha^\dagger)\rho(c_\alpha)w = \rho(1)w - e = w - e \notin V_*.$$

This in turn implies that $\rho(c_\alpha)w \notin V_*$. Also in view of equation (10), we have $\rho(\Pi)\rho(c_\alpha)w = \rho(c_\alpha)w \notin V_*$. This contradicts $\rho(\Pi)(\rho(c_\alpha)w) \in V_0 \subset V_*$. Therefore, such a w does not exist and $V = V_*$. □

Next consider the case where V is endowed with an inner product $\langle\ |\ \rangle$. Then we have the following results.

Lemma 5. *Let e_{i_1} and e_{i_2} be orthogonal basis vectors of V_0. Then V_{i_1} and V_{i_2} are orthogonal subspaces of V.*

Proof. This statement follows from the identities

$$\langle\rho(c_\alpha^\dagger)e_{i_1}|\rho(c_\beta^\dagger)e_{i_2}\rangle = \langle e_{i_1}|\rho(c_\alpha c_\beta^\dagger)e_{i_2}\rangle = \delta_{\alpha\beta}\langle e_{i_1}|\rho(\Pi)|e_{i_2}\rangle = \delta_{\alpha\beta}\langle e_{i_1}|e_{i_2}\rangle = 0$$

$$\langle e_{i_1}|\rho(c_\alpha^\dagger)e_{i_2}\rangle = \langle\rho(c_\alpha)e_{i_1}|e_{i_2}\rangle = 0$$

where we have made use of equation (19) and $\rho(c_\alpha)e_{i_1} = 0$. □

This lemma implies that V_* is actually an orthogonal direct sum of the subspaces V_i.

Lemma 6. *Let $V_*^c := \{w \in V | \forall v \in V_*, \langle w|v\rangle = 0\}$ be the orthogonal complement of V_*. Then V_*^c is the representation space for a trivial representation of $\mathcal{A}$.*

Proof. Let $w \in V_*^c$ and $e := \rho(\Pi)w \in V_0 \subset V_*$. Then clearly, $\langle w|e\rangle = 0$ and

$$\langle e|e\rangle = \langle\rho(\Pi)w|\rho(\Pi)w\rangle = \langle w|\rho(\Pi)^\dagger\rho(\Pi)w\rangle = \langle w|\rho(\Pi)w\rangle = \langle w|e\rangle = 0. \tag{21}$$

Here, we have made use of equations (9) and (19). Equation (21) implies $\rho(\Pi)w = e = 0$. Hence $\rho(\Pi)(V_*^c) = \{0\}$. Now, computing

$$\langle\rho(c_\alpha^\dagger)w|\rho(c_\alpha^\dagger)w\rangle = \langle w|\rho(c_\alpha)\rho(c_\alpha^\dagger)w\rangle = \langle w|\rho(\Pi)w\rangle = 0$$

we find that

$$\rho(c_\alpha^\dagger)w = 0. \tag{22}$$

Next, using equation (7), we have

$$\left[\rho(1) - \sum_{\alpha=1}^{p}\rho(c_\alpha^\dagger)\rho(c_\alpha)\right]w = \rho(\Pi)w = 0. \tag{23}$$

Furthermore, let us express $\rho(c_\alpha)w = w_\alpha + w_\alpha^c$ where $w_\alpha \in V_*$ and $w_\alpha^c \in V_*^c$. Then according to the above argument $\rho(c_\alpha^\dagger)w_\alpha^c = 0$. This together with equations (22) and (23) imply

$$\begin{aligned}\langle\rho(1)w|\rho(1)w\rangle &= \langle w|\rho(1^\dagger)\rho(1)w\rangle = \langle w|\rho(1)w\rangle \\ &= \sum_{\alpha=1}^{p}\langle w|\rho(c_\alpha^\dagger)\rho(c_\alpha)w\rangle = \sum_{\alpha=1}^{p}\langle w|\rho(c_\alpha^\dagger)w_\alpha\rangle = 0.\end{aligned} \tag{24}$$

The last equality follows from the fact that since $w_\alpha \in V_*$, $\rho(c_\alpha^\dagger)w_\alpha \in V_*$. Equation (24) implies $\rho(1)w = 0$. Therefore, for all $x \in A$,

$$\rho(x)w = \rho(x1)w = \rho(x)\rho(1)w = 0$$

and V_*^c yields a trivial representation of $\mathcal{A}$. □

In summary, up to equivalence, the orthofermion algebra of order p has a unique nontrivial $(p+1)$-dimensional irreducible representation ρ_0, and every representation decomposes into irreducible representations that are equivalent to either the trivial representation or ρ_0. In particular, the orthosupersymmetry algebra is in a sense the unique generalization of the supersymmetry algebra describing Bose-orthoFermi symmetry.

4. The Ladder operators of the canonical representation

Consider the canonical irreducible representation ρ_0 of section 2 and let

$$L := c_1 + \sum_{\alpha=2}^{p} c_{\alpha-1}^{\dagger} c_{\alpha}. \tag{25}$$

Then, in view of equations (8), (14)–(16), we have

$$\begin{aligned} L|n\rangle &= \begin{cases} 0 & \text{for} \quad n = 0 \\ |n-1\rangle & \text{for} \quad n \in \{1, 2, \ldots, p\} \end{cases} \\ L^{\dagger}|n\rangle &= \begin{cases} |n+1\rangle & \text{for} \quad n \in \{0, 1, \ldots, p-1\} \\ 0 & \text{for} \quad n = p \end{cases}. \end{aligned} \tag{26}$$

These equations show that L and $L^{\dagger}$ are the ladder operators for the canonical representation of the orthofermion algebra.

The ladder operators L and $L^{\dagger}$ have certain interesting properties. For example, we can use equations (25), (8), (10) and (11), to compute

$$L^{\dagger}L = 1 - \Pi \tag{27}$$

$$LL^{\dagger} = 1 - c_p^{\dagger} c_p \tag{28}$$

$$L^k = \begin{cases} c_k + \displaystyle\sum_{\alpha=1}^{p-k} c_{\alpha}^{\dagger} c_{\alpha+k} & \text{for} \quad k \in \{1, 2, \ldots, p-1\} \\ c_p & \text{for} \quad k = p \\ 0 & \text{for} \quad k \in \{p+1, p+2, \ldots\} \end{cases} \tag{29}$$

$$L^p L^{\dagger} = c_{p-1} \qquad L^{\dagger} L^p = L^{p-1} - c_{p-1}. \tag{30}$$

In view of these equations and (10) and (11), we also obtain

$$L^k \Pi = 0 \qquad \Pi L^k = c_k \tag{31}$$

$$L^{p-k} L^{\dagger} L^k = L^{p-1} \tag{32}$$

where $k \in \{1, 2, \ldots, p\}$.

Equations (29)–(32) imply

$$L^{p+1} = 0 \tag{33}$$

$$\sum_{k=0}^{p} L^{p-k} L^{\dagger} L^k = p\, L^{p-1}. \tag{34}$$

These equations are reminiscent of the defining equations for the parasupersymmetry of order p, [4]. In section 5, we shall use these equations to establish that every orthosupersymmetric system of order p has a parasupersymmetry of order p.

Next, let

$$F := L + c_p^{\dagger}. \tag{35}$$

Then, in view of equations (14), (15) and (26), we have

$$F|n\rangle = \begin{cases} |n-1\rangle & \text{for} \quad n \in \{1, \ldots, p\} \\ |p\rangle & \text{for} \quad n = 0 \end{cases}$$
$$F^\dagger|n\rangle = \begin{cases} |n+1\rangle & \text{for} \quad n \in \{0, 1, \ldots, p-1\} \\ |0\rangle & \text{for} \quad n = p. \end{cases} \tag{36}$$

In particular,

$$F^{p+1} = 1. \tag{37}$$

In the following section, we shall make use of this identity to show that every orthosupersymmetric system of order p has a fractional supersymmetry of order p.

5. An orthosupersymmetric realization of parasupersymmetry and fractional supersymmetry

In [2], the algebra (1)–(3) is used to show that the operator

$$\tilde{Q} := Q_1^\dagger + \sum_{\alpha=2}^{p} Q_\alpha^\dagger Q_{\alpha-1} + Q_p$$

satisfies $\tilde{Q}^{p+1} = (2H)^p$. Therefore, $\tilde{Q}$ is the generator of a fractional supersymmetry for the Hamiltonian $K := (2H)^p$. In this section, we shall demonstrate that any orthosupersymmetric system has a fractional supersymmetry and a parasupersymmetry of order $p + 1$.

First, we recall that the energy spectrum of an orthosupersymmetric Hamiltonian H is non-negative. This follows from equation (2). Setting $\alpha = \beta$ in this equation, we have for any state vector $|\psi\rangle$,

$$\langle\psi|H|\psi\rangle = ||Q_\alpha^\dagger|\psi\rangle||^2 + \sum_{\gamma=1}^{p} ||Q_\gamma|\psi\rangle||^2 \geqslant 0. \tag{38}$$

Next, Let E denote an eigenvalue of H and $\mathcal{H}^{(E)}$ denote the corresponding eigenspace. Because of equation (1) the restriction $Q_1^{(E)} := Q|_{\mathcal{H}^{(E)}}$ is an operator mapping $\mathcal{H}^{(E)}$ into $\mathcal{H}^{(E)}$. Restricting equations (2) and (3) to $\mathcal{H}^{(E)}$, we find

$$Q_\alpha^{(E)} Q_\beta^{(E)\dagger} + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} Q_\gamma^{(E)\dagger} Q_\gamma^{(E)} = 2\delta_{\alpha\beta} E\, I^{(E)} \tag{39}$$

$$Q_\alpha^{(E)} Q_\beta^{(E)} = 0 \tag{40}$$

where $I^{(E)}$ denotes the identity operator on $\mathcal{H}^{(E)}$.

Now, if $E = 0$, then according to equation (38), we have

$$Q_\alpha^{(0)} = Q_\alpha^{(0)\dagger} = 0. \tag{41}$$

Next, introduce

$$c_\alpha^{(E)} := \begin{cases} 0 & \text{for} \quad E = 0 \\ (2E)^{-1/2} Q_\alpha^{(E)} & \text{for} \quad E > 0. \end{cases} \tag{42}$$

Then in terms of $c_\alpha^{(E)}$, equations (39) and (40), take the form

$$c_\alpha^{(E)} c_\beta^{(E)^\dagger} + \delta_{\alpha\beta} \sum_{\gamma=1}^{p} c_\gamma^{(E)\dagger} c_\gamma^{(E)} = \begin{cases} 0 & \text{for} \quad E = 0 \\ 2\delta_{\alpha\beta} I^{(E)} & \text{for} \quad E > 0 \end{cases} \tag{43}$$

$$c_\alpha^{(E)} c_\beta^{(E)} = 0. \tag{44}$$

Comparing these equations with equations (4) and (5), we see that $c_\alpha^{(E)}$ provide a representation $\rho^{(E)}$ of the orthofermion algebra,

$$c_\alpha^{(E)} = \rho^{(E)}(c_\alpha). \tag{45}$$

Clearly, for $E = 0$, this representation is the direct sum of a number $n_0 = \dim(\mathcal{H}^{(0)})$ of trivial representations. Moreover, in view of equation (43), for $E > 0$, the identity operator 1 is represented by $I^{(E)}$. Therefore, according to lemmas 4 and 5, the representation $\rho^{(E)}$ decomposes into a number n_E of irreducible representations $\rho_i^{(E)}$ which are equivalent to the canonical representation ρ_0. Denoting the corresponding representation spaces by $\mathcal{H}_i^{(E)}$, we can express $\mathcal{H}^{(E)}$ as an orthogonal direct sum of $\mathcal{H}_i^{(E)}$,

$$\mathcal{H}^{(E)} = \oplus_{i=1}^{n_E} \mathcal{H}_i^{(E)}.$$

A direct implication of the fact that $c_\alpha^{(E)}$ provide a representation $\rho^{(E)}$ which in turn decomposes into the irreducible representations $\rho_i^{(E)}$ is that the positive energy eigenvalues E are $n_i(p+1)$-fold degenerate. This confirms the results of [2] on the topological symmetries [3] of orthosupersymmetric systems.

Next, let $L^{(E)} := \rho^{(E)}(L)$ and $F^{(E)} := \rho^{(E)}(F)$, where L and F are the operators introduced in equations (25) and (35), respectively. In view of the above mentioned decomposition of $\rho^{(E)}$ into $\rho_i^{(E)}$, the equivalence of the latter with ρ_0 and equations (33), (34) and (37), we have

$$(L^{(E)})^{p+1} = 0 \qquad \sum_{K=0}^{p} (L^{(E)})^{p-k} L^{(E)\dagger} (L^{(E)})^k = p\,(L^{(E)})^{p-1} \tag{46}$$

$$(F^{(E)})^{p+1} = 1. \tag{47}$$

Now, consider the operators Q and $\mathcal{Q}$ defined through their restrictions $Q^{(E)}$ and $\mathcal{Q}^{(E)}$ on the eigenspaces $\mathcal{H}^{(E)}$ according to

$$Q^{(E)} := \begin{cases} 0 & \text{for} \quad E = 0 \\ \sqrt{2E}\, L^{(E)} & \text{for} \quad E > 0. \end{cases} \tag{48}$$

$$\mathcal{Q}^{(E)} := \begin{cases} 0 & \text{for} \quad E = 0 \\ E^{1/(p+1)} F^{(E)} & \text{for} \quad E > 0. \end{cases} \tag{49}$$

Then, in view of equations (46) and (47), we have

$$Q^{p+1} = 0 \qquad \sum_{k=0}^{p} Q^{p-k} Q^\dagger Q^k = p\, Q^{p-1} H \tag{50}$$

$$\mathcal{Q}^{p+1} = H. \tag{51}$$

Furthermore, by construction,

$$[Q, H] = [\mathcal{Q}, H] = 0.$$

These equations indicate that the system has a parasupersymmetry [4] of order p generated by Q and a fractional supersymmetry [5] of order $p + 1$ generated by $\mathcal{Q}$.

Note also that the parasupersymmetry and fractional supersymmetry generators can be expressed in terms of the orthosupersymmetry generators Q_α according to

$$Q = Q_1 + (2H)^{-1/2} \sum_{\alpha=2}^{p} Q_{\alpha-1}^\dagger Q_\alpha$$

$$\mathcal{Q} = 2^{-1/2} H^{-\frac{p-1}{p+1}} Q_1 + 2^{-1} H^{-\frac{p}{p+1}} \sum_{\alpha=2}^{p} Q_{\alpha-1}^\dagger Q_\alpha + 2^{-1/2} H^{-\frac{p-1}{p+1}} Q_p^\dagger.$$

Here, for all $a \in \mathbb{R}^+$, $H^a := \sum_E E^a \Lambda_E$ and Λ_E is the projection operator onto the eigenspace $\mathcal{H}^{(E)}$.

6. Summary and conclusion

In this paper, we addressed the representation theory of orthofermions. We constructed a canonical $(p + 1)$-dimensional irreducible representation for the orthofermion algebra of order p, and showed that every representation of this algebra decomposes into copies of the trivial and the canonical representation. The canonical representation which is a Fock space representation admits ladder operators. We obtained these ladder operators and their properties to establish parasupersymmetry and fractional supersymmetry of general orthosupersymmetric Hamiltonians. Our results may be viewed as a novel realization of parasupersymmetry and fractional supersymmetry of arbitrary order. In a sense, it yields an alternative statistical interpretation of these symmetries.

As argued in [2] and shown in this paper, orthosupersymmetric systems satisfy the defining properties of certain topological symmetries [3]. The latter are a class of generalizations of supersymmetry that involve topological invariants similar to the Witten index. A proper understanding of these invariants requires the study of concrete toy models displaying these symmetries. The orthosupersymmetric systems provide a class of these models. Our analysis of orthofermion algebra leads to a clear picture of the general properties of orthosupersymmetry in one dimension. A logical extension of our results would be to treat orthofermions and orthosupersymmetry in higher dimensions. This might also shed some light on fractional supersymmetry in higher dimensions.

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